

1. (a) The number of generators in $\phi(U(n))$ is $\phi(\phi(n))$ where $\phi(n)$ is the number of positive integers less than n .

(b)

$$a = 2^{k-1} + 1, b = 2^k - 1$$

$$U(2^k) = \{1, 3, 5, \dots, 2^k - 3, 2^k - 1\}$$

$$(2^{k-1} + 1)^2 = 2^{2k-2} + 1 + 2 \cdot 2^{k-1} = 2^{k-2} 2^k + 2^k + 1 \equiv 1 \pmod{2^k}$$

$$(2^k - 1)^2 = 2^{2k} + 1 - 2 \cdot 2^k = 2^k \cdot 2^k - 2 \cdot 2^k + 1 \equiv 1 \pmod{2^k}$$

$$\{2^{k-1} + 1, 1\}, \{2^k - 1, 1\}$$

These two are distinct elements of order 2.

$$(c) \phi(U(15)) = \{1, 2, 4, 7, 8, 11, 13, 14\}$$

$$\phi(U(21)) = \{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$$

$$\begin{aligned} 2 (a) [r^i, r^j] &= r^i \cdot r^j \cdot (r^i)^{-1} (r^j)^{-1} \\ &= r^{i+j-i-j+2} \\ &= r^{2n} = (r^n)^2 = e^2 = e \end{aligned}$$

$$\begin{aligned} (b) [fr^i, fr^j] &= (fr^i)(fr^j)(fr^i)^{-1}(fr^j)^{-1} \\ &= (fr^i)(fr^j)(fr^i)(fr^j) \\ &= f(r^i f)(r^j f)(r^i f) r^j \\ &= f(fr^{n-i})(fr^{n-j})(fr^{n-i}) r^i \\ &= (f \cdot f)(r^{n-i} f)(r^{n-j} f) r^{n-i+j} \\ &= e \cdot r^{2(j-i)} \\ &= r^{2(j-i)} \end{aligned}$$

$$\begin{aligned} (c) [r^i, fr^j] &= r^i(fr^j)(r^i)^{-1}(fr^j)^{-1} \\ &= r^i(fr^j)r^{n-i}(fr^j) \\ &= (r^i f)(r^{n-i+j} f)r^j \\ &= r^i(f \cdot f)r^{i-j} \cdot r^j \\ &= r^{i+1-j+j} = r^{2i} \end{aligned}$$

$$(d) D'_n = \{ r^{2^i} \mid i \in 1, 2, \dots, n \}$$

$$n \text{ is odd } D'_n = \langle r, r^2, \dots, r^n \rangle$$

$$\frac{D}{D'_n} = \langle e, s \rangle + D'_n$$

$$[D'_n : D'_n] = 2$$

$$n \text{ is even } D'_n = \langle r^2, r^4, \dots, r^n \rangle$$

$$\frac{D}{D'_n} = \langle e, s, r, sr \rangle + D'_n$$

$$[D'_n : D'_n] = 4$$

$$(e) Z(D_n) = \begin{cases} \{e\} & ; \text{ if } n \text{ is odd} \\ \{e, r^{n/2}\} & ; \text{ if } n \text{ is even} \end{cases}$$

$$3(p)(i) \quad x, y \in G$$

$$L(x) = \tau(y)$$

$$x^{-1} = y^{-1}$$

$$x^{-1}y = e$$

$$y = x$$

Let $y \in G$, then $y^{-1} \in G$ since $L(y^{-1}) = (y^{-1})^{-1} = y$

$$L \in \text{Sym}(G)$$

$$(ii) o(G) = n, \quad \phi_G(2) = a$$

$$l(g) = g^{-1}, \quad l^2(g) = (g^{-1})^{-1} = g \quad (a+1, n-a-1, 0, \dots, 0)$$

$$(iii) o(G) = 2m, \quad \phi_G(2) = 0 \quad (1, 2n-1, 0, \dots, 0)$$

$$\text{elements of } G \text{ are } 2(2m-1)+1 = 4m-1$$

$$(b) (i) \quad x, y \in G \quad p(x) = p(y)$$

$$gx = gy$$

$$x = y$$

$$\text{Let } y \in G \quad g^{-1}y \in G \text{ since } p(g^{-1}(y)) = g(g^{-1}(y)) = y$$

(ii) $o G(n), o(g) = b \quad x \in G$

$p(x) = g(x), p^2(x) = g^2(x), \dots, p^{b-1}(x) = g^{b-1}(x), \cancel{p^b(x)}$
 $(0, \dots, 0, \dots, \frac{n}{b}, 0, \dots, 0)$

(iii) $\frac{n}{b}$ cycles must have $\frac{n}{b} \in \mathbb{Z}$

4. Partition	Cycle type	Order	number
1 1 1 1 1 1	(1)(2)(3)(4)(5)(6)	1	1
1 1 1 1 2	(1, 2)	2	15
1 1 1 3	(1, 2, 3)	3	40
1 1 4	(1, 2, 3, 4)	4	90
1 1 2 2	(1, 2)(3, 4)	2	45
1 5	(1, 2, 3, 4, 5)	5	144
1 2 3	(1, 2, 3)(4, 5)	6	120
2 2 2	(1, 2)(3, 4)(5, 6)	2	15
4 2	(1, 2, 3, 4)(5, 6)	4	90
3 3	(1, 2, 3)(4, 5, 6)	3	40
6	(1, 2, 3, 4, 5, 6)	6	120

d	1	2	3	4	5	6
$\phi(S_6)$	1					

n	1	2	3	4	5	6
$\exp(S_6)$	1	2	6	12	60	60

$$5a) \sigma = (2, 3)(4, 5, 6)(7, 8, 9, 10) \dots$$

$$b) (1, 2, 3, 4, 5, \dots) \quad \text{or} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ 2 & 3 & 4 & 5 & 6 & 7 & \dots \end{pmatrix}$$

$$c) (3, 5)(7, 9, 11)(13, 15, 17, 19) \dots$$

$$(2, 4, 6, 8, 10, \dots)$$

$$d) (1, 2, 2^2, 2^3, 2^4, \dots)$$

$$(3, 3^2, 3^3, 3^4, \dots)$$

$$(5, 5^2, 5^3, \dots)$$